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Rotor Interaction in the Annulus Billiard

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Recent numerical simulations indicate that the Lorentz gas with rotating scatterers yields a good description of transport phenomena. Motivated by this we introduce the rotor interaction in the integrable system of the annulus billiard. We prove the presence of a variety of dynamical behavior, from integrability to ergodicity.

KEY WORDS: Billiards; rotor interaction; integrability; ergodicity; Anzai skew product.

1. INTRODUCTION

The dynamical system of point particles moving freely and colliding elastically with a periodic array of fixed circular scatterers, usually referred to as the Lorentz gas, is one of the most successful models in nonequilibrium statistical mechanics. The model, despite of its popularity, has some obvious limitations. One of the principal drawbacks is that the point particles do not interact, thus their energy is preserved individually, which accounts for a growing number of conserved quantities as the number of degrees of freedom is increased.

To get around this limitation one can consider modifications of the Lorentz gas in which the point particles interact in an indirect way. More precisely, what has been proposed recently in refs. 10, 11, is to introduce further "internal" degrees of freedom corresponding to the scatterers.

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The usual elastic collision is replaced by an interaction which links the individual point particles *via* this new degree of freedom of the scatterers. The model with possibly the most transparent physical interpretation is that of *the Lorentz gas with rotating scatterers (LGRS)*. In this setting, first mentioned in ref. 11 and later intensively studied in refs. 9 and 10, the scatterers rotate with some angular velocity ω. In course of a collision of the point particle with a rotating scatterer, in a way uniquely determined by the conservation of local quantities, both the angular velocity and the point particle velocity changes, see Formula (1.1). We will refer to such an interaction as the *rotor interaction*. A recent detailed numerical investigation of LGRS in ref. 9 has shown that this model may be a very good candidate to describe nonequilibrium phenomena. In contrast to the usual Lorentz gas, issues like local thermal equilibrium or relations of transport coefficients can be addressed in this modified setting.

Witnessing the success of these numerical investigations, one question naturally arises, namely to give a mathematically rigorous discussion of LGRS. This task seems to be quite difficult, so far the only rigorous results obtained give a detailed analysis of a certain zero measure set of trajectories that possess some special symmetries, see ref. 4. Difficulties are mainly related to the fact that the introduced new degrees of freedom implement nonhyperbolic features into the dynamical system in the directions where the rotational and translational degrees of freedom interact. Hyperbolic and nonhyperbolic phenomena mix in the system in a highly nontrivial way.

What we propose in this article is to introduce the same interaction in other billiard systems. The further we are away from the hyperbolic setting, the more transparent the effect of the rotor interaction on the dynamics will become. Thus as a first step we investigate a natural integrable candidate: a point particle in the annular region bounded by two concentric circles, the outer having radius one, the inner radius $R<1$. Interaction with the circular scatterers is either an elastic reflection (as in billiard models) or the scatterer is assumed to rotate with some angular velocity ω and interact with the point particle according to the laws:

$$\begin{aligned} v_{\mathrm{n}}' &= -v_{\mathrm{n}}, \\ v_t' &= v_{\mathrm{t}} - \frac{2\eta}{1+\eta}(v_{\mathrm{t}} - r\omega), \\ r\omega' &= r\omega + \frac{2}{1+\eta}(v_{\mathrm{t}} - r\omega). \end{aligned} \tag{1.1}$$

Here the coordinates v_{n} and v_{t} are the normal and tangential components (with respect to the circle of collision) of the particle velocity and the primes indicate post-collisional values. Furthermore r is 1 for the outer and R for the inner circular scatterer, and $\eta = (\Theta/mr^2)$, where Θ is the moment of inertia for the scatterer in question and m is the mass of the point particle.

The inner scatterer is always assumed to rotate. Furthermore, in Section 3 the outer scatterer is replaced by a rotor as well. We would like to emphasize that collisions of the point particle with rotating scatterers are not the usual elastic reflections (angle of reflection equals angle of incidence), they are the rotor interactions of Formula (1.1).

This geometrical setting has the following advantages which in particular allow for a mathematically complete discussion.

- As mentioned above, the classical billiard system in the annulus is integrable (ref. 12, p. 26). Thus we may have a good understanding of the mechanism of the rotor interaction as it deviates the system from integrability.
- The annulus geometry has the obvious advantage that the scatterers rotate around the same center. This accounts for the presence, in addition to the full kinetic energy, of another conserved quantity: the total angular momentum. These two integrals decrease the dimension of the phase space and thus result in a more transparent setting.

Our article is structured as follows. In Section 2 we give a brief discussion of the system of one point particle in the annulus with the inner scatterer rotating and the outer kept to be a classical billiard. This is a system of two nontrivial degrees of freedom, which we find to remain integrable (Lemma 2.2). This is a natural result in the view of the presence of two nontrivial independent conserved quantities.

However, if we let both scatterers rotate and thus switch on further degrees of freedom, integrability is destroyed. We give a detailed investigation of the system of one point particle with two rotating scatterers in Section 3. First, we discuss evolution of the velocity coordinates (i.e., the coordinates in Formula (1.1)) referred to as *the base* and find that for a positive measure set of parameters this system is minimal (Theorem 3.7). Note that in the classical billiard setting the velocity evolution is constant. Then we turn to the whole system, sometimes referred to as *fibers*, as it is a skew product above the base. In contrast to the integrability of the classical billiard, we prove for a positive measure set of parameters that the whole system is ergodic (Theorem 3.15).

A further feature of our results we would like to emphasize is the following. There is an important physical parameter γ directly related to the rescaled moments of inertia η (cf. Formula (1.1)) with respect to which the dynamical behavior is very sensitive. Roughly speaking, for typical η values, we prove the above mentioned minimality and ergodicity, while for a dense countable exceptional set, velocity evolution is periodic. This sensitivity with respect to η resembles, on the one hand, to similar phenomena that arises in nonhyperbolic dynamical systems like polygonal billiards (see ref. 12). On the other hand, the presence of some exceptional η values for which the system may not be ergodic has also been observed in numerical studies of LGRS (see refs. 9 and 11). Thus this fact provides further evidence that our discussion is relevant to LGRS.

Finally we close our article with some remarks on the system of two particles with one rotating scatterer in Section 4. In contrast to the classical billiard where the motion is a product of the individual integrable systems, the rotor interaction switches on a nontrivial coupling of the two particles. This system is more complicated, thus to keep the length of the paper reasonable we omit a detailed discussion.

2. ONE PARTICLE, ONE ROTATING SCATTERER

This section is devoted to a model which is closest to the original integrable annulus billiard, namely the motion of one point particle with only the inner scatterer rotating. For certain initial conditions the point particle never reaches the inner circular scatterer. In this invariant set motion is that of a point particle in a circle, a system well-known to be integrable. We consider the complement.

At the inner scatterer we have

$$\begin{aligned} v_{\mathrm{n}}' &= -v_{\mathrm{n}}, \\ v_{\mathrm{t}}' &= v_{\mathrm{t}} - \frac{2\eta}{1+\eta}(v_{\mathrm{t}} - R\omega), \\ R\omega' &= R\omega + \frac{2}{1+\eta}(v_{\mathrm{t}} - R\omega), \end{aligned} \tag{2.1}$$

which is just the inner version of Eq. (1.1).

Remark 2.1. Note that for the same velocity vector $\vec{v}$ the splitting into tangential and normal components is different for the inner and the outer scatterer. In the present section and Section 4 it is only the inner scatterer that is assumed to rotate, and thus the components v_{n} and v_{t} always refer to the splitting at the inner scatterer. In Section 3 both scatterers rotate and it is more convenient to use different notation, see the beginning of Section 3.1.

Lemma 2.2. (1) If a trajectory collides once with the inner scatterer, it collides with it infinitely often. More precisely, there is an alternating sequence of collisions, every first with the inner and every second with the outer scatterer.

(2) There are three integrals of motion.

(3) The velocity motion is periodic of period 2.

Proof. A trajectory which starts from the inner circle is reflected elastically in the outer circle, thus it returns to the inner circle. Iterating this argument proves statement (1).

Consider two consecutive reflections at the inner scatterer with one billiard reflection at the outer one in between. Let us denote by the triples $v_{\mathrm{n}}(1), v_{\mathrm{t}}(1), \omega(1)$ and $v_{\mathrm{n}}(2), v_{\mathrm{t}}(2), \omega(2)$ the incoming velocities at the two consecutive inner reflections, and let us refer to the outgoing counterparts by primes. By the geometrical picture sketched above:

$$v_{\mathrm{n}}(2) = -v_{\mathrm{n}}(1)', \qquad v_{\mathrm{t}}(2) = v_{\mathrm{t}}(1)'$$

and furthermore $\omega(1)' = \omega(2)$ as the inner scatterer does not interact while the particle collides outside. This, together with (2.1), gives, on the one hand

$$v_{\mathrm{n}}(1) = v_{\mathrm{n}}(2) = -v_{\mathrm{n}}(1)' = -v_{\mathrm{n}}(2)',$$

which can be regarded as an integral of motion. On the other hand, to get $v_{\mathrm{t}}(2)'$ and $\omega(2)'$ we need to iterate the second and the third formulas in (2.1) twice. This gives

$$\omega(2)' = \omega(1), \qquad v_{\mathrm{t}}(2)' = v_{\mathrm{t}}(1)$$

thus statement (3) is proved.

Finally multiply the third equation in (2.1) with η and add it to the second to get

$$v_{\mathrm{t}} + \eta R\omega = v_{\mathrm{t}}' + \eta R\omega' = N. \tag{2.2}$$

Furthermore, subtracting the third line from the second in (2.1) we get

$$v_{\mathrm{t}}' - R\omega' = -(v_{\mathrm{t}} - R\omega)$$

which, together with (2.2) results in

$$v_t^2+\eta R^2\omega^2=v_t'^2+\eta R^2\omega'^2=E. \tag{2.3}$$

Thus we have calculated the other two integrals. ▮

Remark 2.3. Equation (2.2) is a multiple of the angular momentum, while (2.3) is the "tangential" part of the full kinetic energy (the normal part of the kinetic energy, i.e., the length of v_{n} is in itself conserved as the third integral).

In the space of velocities (v_{t},ω), that can be regarded as $\mathbb{R}^2$, fixing the values (2.2) and (2.3) we get a line and a circle, respectively. These two curves intersect in (at most) two points (v_{t},ω) and $(v_{\mathrm{t}}',\omega')$. Velocity evolution is a period two cycle on these two points.

Turning to the evolution in the fibers, we essentially get the behavior observed in the integrable annulus billiard. Let us parameterize the outer circle by the arclength parameter φ and denote the points of consecutive impacts on a trajectory by $\varphi_k, k\in\mathbb{Z}$. In contrast to the rest of the section, from here on v_{t} and v_{n} are the tangential and normal components *at the outer scatterer*, cf. Remark 2.1. Then

$$\varphi_{k+1}=\varphi_k+\hat\beta+\hat\beta',$$

where

$$\beta=\arctan\frac{v_{\mathrm{t}}}{v_{\mathrm{n}}},\quad \hat\beta=\arcsin\left(\frac{\sin\beta}{R}\right)-\beta$$

and the same for the primed quantities, see Fig. 1. Thus the angles $\hat\beta,\hat\beta'\in[-\arccos(R),\arccos(R)]$ depend smoothly and strictly monotonically on the ratios $(v_{\mathrm{t}}/v_{\mathrm{n}})$ and $(v_{\mathrm{t}}'/v_{\mathrm{n}}')$, respectively. The consecutive φ_k-s follow an orbit of a circle rotation, which is irrational apart from a countable collection of one codimensional manifolds in the three dimensional parameter space of integrals. To see this keep the values of E and N ((2.3) and (2.2)) fixed and increase the value of v_{n} continuously, then $\hat\beta+\hat\beta'$ decreases continuously and is rational only for a countable set of v_{n}-s.

3. ONE PARTICLE, TWO ROTATING SCATTERERS

3.1. Coordinates, Parameters and Integrals of Motion

In this section we will find that allowing both scatterers to rotate results in a behavior different from the integrable models discussed so far.

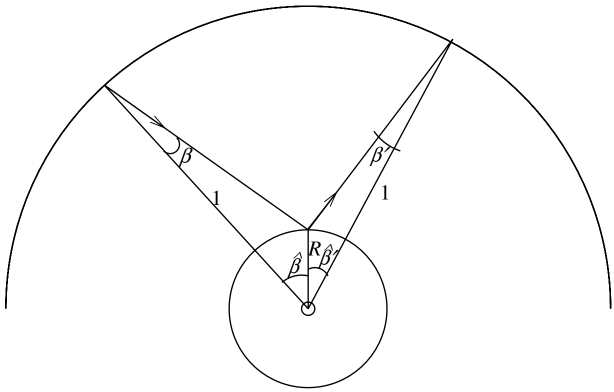

Fig. 1. $\hat{\beta} = \arcsin\left(\frac{\sin\beta}{R}\right) - \beta$.

As long as velocities are concerned we need to handle an *a priori* four-dimensional system: the coordinates are, on the one hand, ω_1, ω_2 for the angular velocities of the inner and outer scatterers, respectively, and, on the other hand, $\vec{v}$, the two dimensional velocity vector of the point particle. Recall Remark 2.1: in contrast to Section 2, we will use notations v_n and v_t for the splitting of the velocity $\vec{v}$ into normal and tangential components at the *outer* scatterer. The sign of v_t has a physical meaning, it describes the direction of the "circular" component of the motion of the point particle. On the contrary, the sign of v_n is in a certain sense irrelevant, the positivity/negativity of v_n merely tells us if we are just before/after a collision with the outer circle, thus we fix v_n to be always positive. The symbols $\bar{v}_\mathrm{n}, \bar{v}_\mathrm{t}$ are used for the splitting of $\vec{v}$ into normal and tangential components at the inner scatterer. By simple geometrical observation they can be defined for vectors $\vec{v}$ which satisfy $(v_\mathrm{t}^2/v_\mathrm{n}^2) \leq R^2/(1-R^2)$ as $\bar{v}_\mathrm{t} = (1/R)v_\mathrm{t}$ and $\bar{v}_\mathrm{n} = \sqrt{v_\mathrm{n}^2 - (1-R^2/R^2)v_\mathrm{t}^2}$. Otherwise the trajectory starting from the outer scatterer reaches it again without interacting with the inner one.

We use the notation ω_i for the angular velocity and Θ_i for the moment of inertia with $i=1,2$ in the case of the inner and the outer scatterer, respectively. The rescaled moments of inertia are $\eta_1 = (\Theta_1/mR^2)$ and $\eta_2 = (\Theta_2/m)$. In these coordinates, the interaction at the outer scatterer is given by:

$$
\begin{aligned}
v_\mathrm{t}' &= v_\mathrm{t} - \frac{2\eta_2}{1+\eta_2}(v_\mathrm{t} - \omega_2), \\
\omega_2' &= \omega_2 + \frac{2}{1+\eta_2}(v_\mathrm{t} - \omega_2).
\end{aligned}
\tag{3.1}
$$

By the convention on the sign of v_n made above, the normal velocity is preserved and straightforward calculation gives two further preserved quantities

$$v_t + \eta_2\omega_2 = v_t' + \eta_2\omega_2', \qquad v_t^2 + \eta_2\omega_2^2 = v_t'^2 + \eta_2\omega_2'^2. \tag{3.2}$$

As to interaction with the inner scatterer

$$\begin{aligned} \bar{v}_t' &= \bar{v}_t - \frac{2\eta_1}{1+\eta_1}(\bar{v}_t - R\omega_1), \\ R\omega_1' &= R\omega_1 + \frac{2}{1+\eta_1}(\bar{v}_t - R\omega_1). \end{aligned}$$

Note, however, that $v_t = R\bar{v}_t$ and thus these equations are equivalent to

$$\begin{aligned} v_t' &= v_t - \frac{2\eta_1}{1+\eta_1}(v_t - R^2\omega_1), \\ R^2\omega_1' &= R^2\omega_1 + \frac{2}{1+\eta_1}(v_t - R^2\omega_1). \end{aligned} \tag{3.3}$$

Two preserved quantities for the inner collisions are

$$v_t + \eta_1 R^2\omega_1 = v_t' + \eta_1 R^2\omega_1', \qquad v_t^2 + \eta_1 R^4\omega_1^2 = v_t'^2 + \eta_1 R^4\omega_1'^2. \tag{3.4}$$

As ω_1 (ω_2) does not change at outer (inner) collisions, (3.2) and (3.4) imply the presence of two integrals of motion, the angular momentum

$$v_t + \eta_1 R^2\omega_1 + \eta_2\omega_2 = \text{const.} = N \tag{3.5}$$

and the tangential energy

$$v_t^2 + \eta_1 R^4\omega_1^2 + \eta_2\omega_2^2 = \text{const.} = E(>0). \tag{3.6}$$

The term "tangential" refers to the fact that the full kinetic energy of the system is a third integral of motion, always greater than the tangential one. More precisely, dividing the full energy $(1/2)(m\vec{v}^2 + \Theta_1\omega_1^2 + \Theta_2\omega_2^2)$ by $(1/2)m$ we get

$$v_n^2 + v_t^2 + \eta_1 R^2\omega_1^2 + \eta_2\omega_2^2 = \text{const.} = F(>E). \tag{3.7}$$

Note that (3.2), (3.4) and (3.7) together imply that v_n ($\bar{v}_n$) is preserved during inner (outer) collisions.

Remark 3.1. (1) One of our aims below is to understand how the dynamical behavior depends on the several parameters of the system. The constants η_1 and η_2 will be referred to as physical parameters in contrast to the integrals of motion N, E and F.

(2) To avoid complications we assume $\vec{v} \neq \vec{0}$. Note that this assumption is reasonable in the following sense: $\vec{v} = \vec{0}$ is only possible for a codimension one submanifold of integrals of motion. To see this observe that if we had $\vec{v} = \vec{0}$, (3.5) and (3.6) would fix the values of the other two coordinates ω_1 and ω_2 (up to two possibilities) and thus would create, via (3.7), a relation between the three integrals of motion.

3.2. Evolution of Velocities

For convenience we rescale our velocities as

$$x := \sqrt{\eta_1} R^2 \omega_1, \qquad y := \sqrt{\eta_2}\omega_2, \quad z := v_{\mathrm{t}}, \quad w := v_{\mathrm{n}}. \tag{3.8}$$

In these rescaled quantities our integrals of motion are

$$x^2 + y^2 + z^2 = E, \qquad \sqrt{\eta_1}x + \sqrt{\eta_2}y + z = N \tag{3.9}$$

and

$$\frac{x^2}{R^2} + y^2 + z^2 + w^2 = F. \tag{3.10}$$

By (3.1) and (3.3) we can first consider how the velocity coordinates x, y, z evolve at collisions. With the convention on its sign the value of $w (= v_{\mathrm{n}})$ is determined by (3.10) for each fixed value of the other three coordinates. In the rest of the section, unless otherwise stated, velocity evolution will always refer to the dynamics in the triple of x, y, z.

Taking into account that x (y) does not change at outer (inner) collisions, in the rescaled coordinates the transformations (3.1) and (3.3) are reflections across the planes

$$\sqrt{\eta_2}z - y = 0 \quad \text{and} \quad \sqrt{\eta_1}z - x = 0, \tag{3.11}$$

respectively. Furthermore, by (3.9) motion is restricted to the intersection of a sphere and a plane, i.e., to a circle which will be referred to as the velocity circle. We will see that the base will consist of two copies of this circle. Note that these circles do not describe the physical position of the particle, on the other hand they give the state of the velocities. On the velocity circle the reflections across the planes (3.11) reduce to reflections across lines. For the outer collisions we have reflections across

$$l_2(t)=\left(-\frac{1+\eta_2}{\sqrt{\eta_1}}t\,,\ \sqrt{\eta_2}t\,,\ N+t\right), \tag{3.12}$$

while for the inner ones across

$$l_1(t)=\left(\sqrt{\eta_1}t\,,\ -\frac{1+\eta_1}{\sqrt{\eta_2}}t\,,\ N+t\right).$$

Let us denote the angle between these two lines by $(\gamma/2)$. Depending on it we distinguish rational, irrational (and later on even Diophantine) cases.

Lemma 3.2. Rationality is independent of the integrals of motion. On the other hand, there exists a countable set of codimension 1 submanifolds of physical parameters which lead to the rational case, otherwise (and thus for a full Lebesgue measure set of parameters) γ is irrational.

Proof. Straightforward calculation gives

$$\cos\frac{\gamma}{2}=\left((1+\eta_1{}^{-1})(1+\eta_2{}^{-1})\right)^{-\frac{1}{2}}. \tag{3.13}$$

Thus the rationality of γ depends only on the physical parameters. Consider the numbers $\cos((p/q)(\pi/2))$ for positive integers $p<q$. Note that elements of this countable set A are all algebraic numbers. The base is rational iff the left hand side of (3.13) is equal to some number in A. This happens for a countable collection of one codimensional submanifolds of physical parameters. ∎

Convention 3.3. To agree with different standard conventions, in the geometric framework we will think of $\mathbb{S}^1$ as $[0,2\pi)$, while in the number theoretic framework we think of $\mathbb{S}^1$ as $[0,1)$. In both cases we scale all $\mathbb{S}^1$ valued quantities $(s,\varphi,\gamma,\alpha(s))$ appropriately. This should cause no confusion.

To give a description of the base dynamics we introduce the arclength parameter s as a coordinate on the velocity circle. For definiteness $s=0$ corresponds to one of the intersection points of the circle with the line

(3.12). In this coordinate the outer and inner bounces, i.e., the transformations (3.1) and (3.3) are

$$s \to s_1 = -s \pmod 1 \text{ and} \tag{3.14}$$
$$s \to s_2 = -s - \gamma \pmod 1, \tag{3.15}$$

respectively. Take two copies, O and I, of the velocity circle describing the outgoing velocities just after the bounces at the inner and outer circles, respectively. The phase space of the velocity motion, or the base, is

$$\hat{M} = \mathbb{S}^1 \times \{1, 2\} = I \cup O, \qquad \hat{M} \ni \bar{s} = (s, i) \quad \text{with } s \in \mathbb{S}^1 \quad \text{and} \quad i = 1, 2.$$

The velocity evolution will be denoted by $T : \hat{M} \to \hat{M}$. As an inner bounce is always followed by an outer one, for $\bar{s} \in I$, i.e. for $(s, 1)$ the image $T\bar{s}$ is always in O, more precisely $T(s, 1) = (s_1, 2)$ (recall (3.14)). An outer bounce, however, may be followed by either an inner or an outer bounce, depending on the angle the point particle velocity makes with the normal vector of the outer scatterer.

Lemma 3.4. (1) There is a (possibly empty) open set $\bar{U} = U \times \{2\} \subset O$, such that for $\bar{s} = (s, 2) \in \bar{U}$, we have $T\bar{s} = (s_1, 2)$ (the image is the point defined by (3.14) in O), and for $\bar{s} \in O \setminus \bar{U}$ we have $T\bar{s} = (s_2, 1)$ (the image is the point defined by (3.15) in I).

(2) The set $U \subset \mathbb{S}^1$ is invariant with respect to the reflection (3.15).

(3) There is an open set of integrals of motion for which U (and consequently $\bar{U}$) is empty.

Remark 3.5. We will refer to empty U as the alternating case, non-empty U as the non-alternating case. Both of them occur for open sets of integrals of motion, independently of the value of physical parameters.

Proof of Lemma 3.4. (1) By Formulas (3.9) and (3.10) the arc-length s on the velocity circle determines $z = v_{\mathrm{t}}$ and $w = v_{\mathrm{n}} \geqslant 0$ continuously. The geometric condition for $(s, 2) = \bar{s} \in O$ to reach the outer scatterer without touching the inner one (and thus interacting with it) is $(v_{\mathrm{t}}^2/v_{\mathrm{n}}^2) > R^2/(1 - R^2)$. Note that at tangential collisions with the inner scatterer v_{t} is modified by (3.3), thus this corresponds to landing on I. We define

$$U := \left\{ s \in \mathbb{S}^1 \mid z^2(s) > \frac{R^2}{1 - R^2} w^2(s) \right\}. \tag{3.16}$$

If $\bar{s} \in \bar{U}$ (i.e., in case the next bounce is outside) then (3.14) applies, and thus $T\bar{s} = (s_1, 2)$. Otherwise (i.e., if $\bar{s} \in O \backslash \bar{U}$) the next bounce is inside, thus the image is the point defined by (3.15) in I, that is $T\bar{s} = (s_2, 1)$. The set $U \subset \mathbb{S}^1$ is the preimage of an open set by a continuous map implicitly defined in Formula (3.16), thus it is open.

(2) To see that U is invariant under (3.15) note first that y does not change during an inner collision. This, together with the presence of the integrals (3.9) and (3.10) implies that $w^2 - ((1-R^2)/R^2)z^2$ is invariant for inner bounces, i.e., for the reflections (3.15). The statement follows as it is exactly the sign of this quantity that distinguishes the points of U.

(3) The set U is empty if

$$z^2(s) < \frac{R^2}{1-R^2} w^2(s) \tag{3.17}$$

for all s. Consider the integrals of motion defined by (3.9) and (3.10) and denote $\lambda = (F/E) > 1$. By (3.9) x^2 and z^2 are both less than or equal to E for all s. This, together with (3.10) gives

$$\begin{aligned} w^2 &\geqslant F - y^2 - z^2 - x^2 - x^2(R^{-2} - 1) \geqslant F - E - (R^{-2} - 1)E = \\ &= F - ER^{-2} = E(\lambda - R^{-2}) \geqslant (\lambda - R^{-2})z^2 \end{aligned}$$

for all s. This implies U is empty if $\lambda - R^{-2} > (1-R^2)/R^2$, i.e., if $\lambda > ((2/R^2) - 1)$, a condition satisfied for a set of parameters that has nonempty interior. ∎

Remark 3.6. (1) Actually U is empty iff we replace the strict inequality in Formula (3.17) with a nonstrict one. The analysis of the equality case is substantially more complicated, and is a codimension one phenomenon in the set of parameters, thus we do not analyze it.

(2) Note that the points $s \in U$ correspond to velocity configurations that cannot be realized at the inner circle. Thus for nonempty U we should modify the definition of I by removing the points corresponding to U, i.e.

$$I = (\mathbb{S}^1 \setminus U) \times \{1\}$$

and the phase space is $M \subset \hat{M}$:

$$M = I \cup O = \Big((\mathbb{S}^1 \setminus U) \times \{1\}\Big) \cup \Big(\mathbb{S}^1 \times \{2\}\Big).$$

Furthermore $T(O\setminus\bar{U})=I$ and $T\bar{U}\cup TI=O$; with

$$\begin{aligned} T(s,2) &= (s_2,1) \quad \text{for} \quad (s,2)\in O\setminus\bar{U},\\ T(s,2) &= (s_1,2) \quad \text{for} \quad (s,2)\in\bar{U},\\ T(s,1) &= (s_1,2) \quad \text{for} \quad (s,1)\in I, \end{aligned} \tag{3.18}$$

where s_1, s_2 are given by Equations (3.14) and (3.15). This is well-defined and invertible by statement (2) of Lemma 3.4.

The main result on the base dynamics is the following theorem.

Theorem 3.7. In the alternating irrational case the dynamics of the base is minimal, while in all the other cases velocity motion is always periodic. Moreover, all possible periods are even with length bounded above by a constant (which depends on the parameters).

Proof. Consider the alternating case first. For $\bar{s}\in O$ all odd iterates are in I and all even iterates are in O (and for $\bar{s}\in I$ the other way round). Thus it is enough to consider the first return map onto O (which is exactly T^2). By (3.14) and (3.15) this is a rotation of $\mathbb{S}^1$ by γ, which is periodic/minimal exactly in the rational/irrational cases. (The odd iterates make the reflected image of the same rotation orbit in I.)

Turning to the nonalternating case, i.e., to $\bar{U}\neq\emptyset$, first note that $\bar{U}\cap T^{-1}\bar{U}$ is an invariant set. To see this consider $(s,2)=\bar{s}\in\bar{U}\cap T^{-1}\bar{U}$. Then both $T\bar{s}$ and $T^2\bar{s}$ are in O. More precisely

$$T(s,2)=(s_1,2) \quad \text{and} \quad T^2(s,2)=T(s_1,2)=(s,2),$$

thus the set is invariant and all points of it are periodic of period 2.

Now take $\bar{s}\in\bar{U}\setminus T^{-1}\bar{U}$. Then $T\bar{s}$ is in $O\setminus\bar{U}$, $T^2\bar{s}\in I$ and even $T^{-1}\bar{s}\in I$, as the point is assumed not to belong to the above described invariant set. Moreover, by (3.14) and (3.18)

$$T\bar{s}=T(s,2)=(s_1,2) \quad \text{and} \quad T^{-1}\bar{s}=T^{-1}(s,2)=(s_1,1), \tag{3.19}$$

thus the trajectory essentially "turns back". The time reflection symmetry of (3.19) is preserved until the next $\bar{U}$-hitting time. To see this let $k>1$ be the smallest integer (possibly ∞) such that $T^k\bar{s}\in\bar{U}$. Up to time k the orbit is alternating, i.e., by induction

$$\begin{aligned} T^{2l-1}\bar{s}&=(s',2), \qquad & T^{2l}\bar{s}&=(s_2',1),\\ T^{-(2l-1)}\bar{s}&=(s',1), \qquad & T^{-2l}\bar{s}&=(s_2',2) \end{aligned}$$

for some $s' \in \mathbb{S}^1 \setminus U$ whenever $0 < 2l < k$. In particular $T^{(k-1)}\bar{s} = (s'', 1)$ and $T^{-(k-1)}\bar{s} = (s'', 2)$ for some $(s'', 2) \in T^{-1}\bar{U} \setminus \bar{U}$ (note that k is always odd). This implies by (3.18) that the preimage of $T^{-(k-1)}\bar{s}$ is not in I, alternation ceases and

$$T^{-k}\bar{s} = (s_1'', 2) \in \bar{U}.$$

Moreover

$$T^k\bar{s} = (s_1'', 2) = T^{-k}\bar{s},$$

thus the point is periodic with period $2k$. This way we have shown that any orbit which enters $\bar{U}$ twice is periodic.

Consider any $\bar{s} = (s, 2) \in O$ and the subset of $\mathbb{S}^1$ generated from s by the reflections (3.14) and (3.15). This subset of $\mathbb{S}^1$ consists of two rotation orbits: the first by γ and the second by $-\gamma$, which is the reflected image of the first one by (3.15).

In the nonalternating rational case there are two possibilities. On the one hand, if both of the above mentioned two rotation orbits avoid U, dynamics is that of the alternating case: $\bar{s}$ is periodic of length that depends only on γ (i.e., on the physical parameters). On the other hand, by the invariance of U with respect to (3.15) (cf. statement (2) of Lemma 3.4) if one of the rotation orbits hit U, the other orbit—the reflected image—should hit it as well. Thus if this occurs the trajectory of $\bar{s}$ enters $\bar{U}$ twice and becomes periodic with even length. Since the longest possible period occurs when the rotational orbit avoids U, the length of all possible periods is bounded above by a constant that depends only on γ.

In the nonalternating irrational case, as both rotation orbits are dense and thus cannot avoid U, the second of the two possibilities sketched above occurs for all $\bar{s}$. To complete the proof of the theorem we only need to show that the length of possible periods is bounded above. By minimality and compactness there exists a positive integer N such that the first N rotation-preimages of U cover $\mathbb{S}^1$. This implies that the length of any period is less than or equal to $4N$. Moreover, N depends only on γ and U, i.e. on the physical parameters and the integrals of motion. ∎

3.3. Analysis of the Skew Product

In this subsection we extend our investigation to the fibers, i.e., to the question how the points of impact for the consecutive outer bounces follow each other on the outer scatterer. Let us parameterize the outer circle by the arclength parameter φ (the starting point is irrelevant and may be chosen arbitrarily). We consider the skew product

$$F:\mathbb{S}^1\times\mathbb{S}^1\to\mathbb{S}^1\times\mathbb{S}^1, \qquad F(s,\varphi)=(T_O s,\varphi+\alpha(s)). \tag{3.20}$$

Here T_O is the (projection onto $\mathbb{S}^1$ of the) first return map onto O for T from Section 3.2. Note that T_O is either the first or the second iterate of T. More precisely, if $s\in U$, we have $T_O=T$ and $\alpha(s)=\pi-2\beta(s)$ where $\tan\beta(s)=v_{\rm t}(s)/v_{\rm n}(s)=z(s)/w(s)$. On the other hand, if $s\in\mathbb{S}^1\setminus U$, we have $T_O=T^2$ and $\alpha(s)$ can be calculated using the geometry of Fig. 1 (cf. the end of Section 2) as

$$\alpha(s)=\hat{\beta}(s)+\hat{\beta}(s_2) \tag{3.21}$$

where

$$\sin(\beta(s)+\hat{\beta}(s)) = \frac{\sin\beta(s)}{R},$$
$$\tan\beta(s) = \frac{v_{\rm t}(s)}{v_{\rm n}(s)}=\frac{z(s)}{w(s)}$$

and s_2 is defined by (3.15). Note that, when restricted to U or $\mathbb{S}^1\setminus U$, both α and T_O are continuous.

Recall from Theorem 3.7 that apart from the alternating irrational case T_O is periodic for any s with some period $N=N(s)$. In these periodic situations

$$F^N(s,\varphi)=(s,\varphi+\alpha_N(s)), \qquad \alpha_N(s)=\alpha(s)+\alpha(T_O s)+\cdots+\alpha({T_O}^{N-1}s).$$

Thus a suitable iterate of the dynamics in the fibers is a rotation, the angle of which depends only on the initial velocity s, and not on the initial position φ. The iterates $F^{k_0+kN}s$, with $0<k_0<N$ fixed and $k\in\mathbb{Z}$, are the shifted versions of the same rotation orbit. Moreover, by piecewise continuity of α and T_O, and by the uniform upper bound on $N(s)$, the rotation angle $\alpha_N(s)$ depends on s in a piecewise continuous manner as well (at discontinuity points, however, even $N(s)$ may jump).

Remark 3.8. Note that in the piecewise continuity statements we have finitely many pieces since the set U,being defined by algebraic equations and inequalities, has finitely many connected components.

From this point on we restrict to the alternating irrational case. Thus T_O is an irrational rotation, and we have

$$F(s,\varphi)=(s+\gamma,\varphi+\alpha(s)) \tag{3.22}$$

with γ irrational and $\alpha:\mathbb{S}^1\to\mathbb{S}^1$ continuous. Such systems are called Anzai skew products after their first investigator (ref. 1). There is an extensive literature on the subject, see refs. 2 and 5–7 and references therein. Some simple issues are even treated in general monographs on dynamical systems like ref. 8.

In addition to the skew product map (3.22) we are interested in the dynamical behavior in real time, i.e., in a suspension flow over our skew product. The ceiling function for this flow, i.e., the physical (real-valued) first return time to the outer scatterer depends only on the initial velocity, thus it may be denoted as $\tau(s)$ with $\tau:\mathbb{S}^1\to\mathbb{R}^+$. We may calculate $\tau(s)$ by Figure 2 as

$$\tau(s)=\frac{d(s)}{|\vec{v}(s)|}+\frac{d(s_2)}{|\vec{v}(s_2)|} \tag{3.23}$$

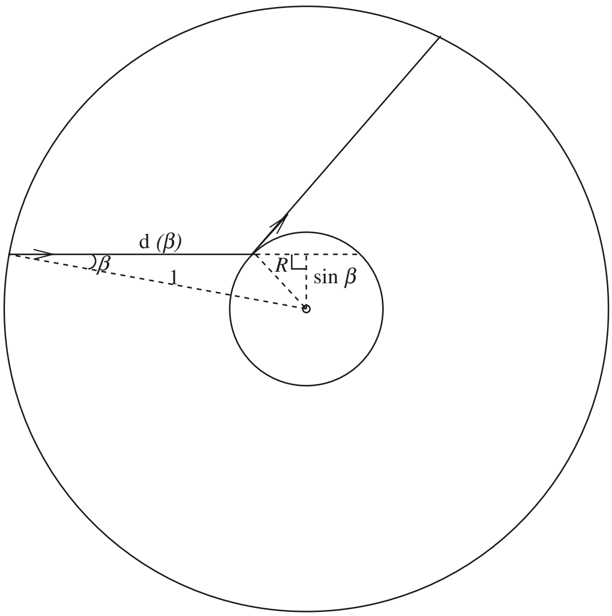

Fig. 2. $d(\beta)=\cos\beta-\sqrt{R^2-\sin^2\beta}$.

with

$$|\vec{v}(s)| = \sqrt{v_t(s)^2 + v_n(s)^2} = \sqrt{z(s)^2 + w(s)^2}$$

and

$$d(\beta(s)) = d(s) = \cos\beta(s) - \sqrt{R^2 - \sin^2\beta(s)},$$

where

$$\tan\beta(s) = \frac{v_t(s)}{v_n(s)} = \frac{z(s)}{w(s)}$$

and s_2 is defined by (3.15).

We recall some basic concepts and properties related to irrational numbers γ and continuous maps $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ (on details see refs. 3 and 8).

Definition 3.9. Given $r > 1$ an irrational number $\gamma \in [0, 1]$ is **Diophantine** of type r if there exists $c > 0$ such that for any $p, q \in \mathbb{Z}$, $q \neq 0$

$$|q\gamma - p| > cq^{-r}.$$

A number which satisfies the above condition for some $r > 1$ is called Diophantine. Liouville numbers γ are those irrationals that do not satisfy the Diophantine condition for any r. It is not hard to see that the set of Diophantine numbers of type r is, as a subset of [0, 1], on the one hand, of the first category, and, on the other hand, of full Lebesgue measure, with, even more, its complement having Hausdorff dimension r^{-1}. Consequently, the set of Diophantine numbers is of full measure and first category, while its complement, the union of Liouville numbers and rational ones, has zero Hausdorff dimension.

Definition 3.10. For any continuous map $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ there exist countably many continuous functions $A : \mathbb{R} \to \mathbb{R}$ such that $A(s+1) - A(s)$ is an integer independent of s and $\alpha(s)$ can be regarded as the restriction of $A(s)$ to $[0, 1)$. The integer $A(s+1) - A(s)$ is independent of the lift and is called **the degree** of $\alpha(s)$. The map $\alpha(s)$ can be regarded as a real-valued continuous function defined on $\mathbb{S}^1$ iff its degree is zero.

Definition 3.11. Two measurable maps $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ and $\alpha' : \mathbb{S}^1 \to \mathbb{S}^1$ are said to be **cohomologous** (for the rotation by γ) iff there exists $V : \mathbb{S}^1 \to \mathbb{S}^1$ measurable such that

$$\alpha(s) = \alpha'(s) + V(s+\gamma) - V(s)$$

for almost all $s \in \mathbb{S}^1$. Maps cohomologous to 0 are called coboundaries.

The following statements date back to ref. 1.

Lemma 3.12. (1) The skew product (3.22) is ergodic (with respect to Lebesgue measure) iff there is no $p \in \mathbb{Z} \setminus \{0\}$ such that $p\alpha(s)$ is a coboundary.

(2) The skew product (3.22) has pure point spectrum iff $\alpha(s)$ is not cohomologous to a constant.

(3) A continuous map $\alpha(s) = \alpha$ is a coboundary iff α and γ are rationally dependent.

(4) By (1-3) for a function $\alpha(s)$ cohomologous to a constant α the skew product is either not ergodic or ergodic with pure point spectrum, depending on whether α and γ are rationally dependent or not.

The following result is a consequence of Theorem 1 from ref. 2 combined with Lemma 3.12.

Lemma 3.13. Consider an Anzai skew product (3.22) with γ Diophantine and $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ of zero degree and C^∞. Then there exists a number $\alpha \in [0, 1)$ such that the map F is smoothly (i.e., C^∞) conjugate to

$$F_0(\mathbb{S}^1 \times \mathbb{S}^1) \to (\mathbb{S}^1 \times \mathbb{S}^1), \qquad F_0(s, \varphi) = (s + \gamma, \varphi + \alpha). \tag{3.24}$$

We prepare for our main result related to the full system, Theorem 3.15, with the proof of the following Lemma.

Lemma 3.14. (1) The map $\alpha(s)$ is of zero degree, thus can be regarded as $\alpha : \mathbb{S}^1 \to \mathbb{R}$.

(2) Both $\alpha(s)$ and $\tau(s)$ are C^∞.

Proof. (1) Note that any continuous map $\mathbb{S}^1 \to \mathbb{S}^1$ that has nonzero degree must be onto. On the other hand, as we are in the alternating case we have $\cos \hat{\beta}(s) > R$ for any $s \in \mathbb{S}^1$ (see Figure 1). This means $|\alpha(s)| \leq 2\arccos R$ which in particular implies $\alpha(s)$ cannot be onto, thus it is of zero degree.

(2) The maps $\alpha(s)$ and $\tau(s)$ are defined in Formulas (3.21) and (3.23) respectively (see also (3.15)). To establish the desired smoothness properties we need to show that four functions, namely:

(i) $\beta(s)$; (ii) $\vec{v}(s)$; (iii) $\hat{\beta}(\beta)$; (iv) $d(\beta)$ are all smooth. Moreover we need to show that (v) $|\vec{v}(s)|$ is bounded away from zero.

First note x, y, z all depend on s smoothly, while w a priori only continuously. To see this note that $x(s), y(s), z(s)$ are the Cartesian coordinates of a circle (of the velocity circle, cf. Section 3.2) thus depend smoothly on the arclength parameter. By (3.10) $w(s)$ is the square root of a smooth nonnegative function, thus it is continuous, and smooth if bounded away from zero, a fact we establish below.

By Remark 3.1 (2) and compactness $|\vec{v}(s)|$ is bounded away from zero, thus we have statement (v). This by alternating (cf. (3.17)) gives positive lower bound and, by the above reasoning, smoothness for $w(s)$ which means (cf. (3.8)) we have shown (ii). Moreover (i) follows as we have demonstrated that $\tan\beta(s)$ is smooth.

We need to prove (iii) and (iv). As we are in the alternating case, see the first part of Remark 3.6, we have Formula (3.17) and thus by compactness

$$R^2 - \sin^2\beta(s) > K_1 > 0 \qquad \text{and} \qquad \frac{\sin^2\beta}{R^2} < K_2 < 1.$$

This means that the quantity under the square root in the formula for $d(\beta)$ (cf. (3.23)) is bounded away from zero. Similarly, in the formula for $\hat{\beta}(\beta)$ (see (3.21)), the argument of arc sin is bounded away from 1 and -1. Thus we have (iii) and (iv). ∎

Theorem 3.15. Let us fix γ Diophantine. (1) The map (3.22) is smoothly conjugate to (3.24) with $\alpha = \int_{\mathbb{S}^1} \alpha(s)ds$.

(2) There is a positive measure set of integrals of motion for which (3.22) is uniquely ergodic.

(3) The system in "physical" time, i.e., the suspension flow over (3.22) with ceiling function (3.23) is never ergodic.

Proof. (1) Lemmas 3.14 and 3.13 imply that (3.22) is smoothly conjugate to (3.24).

(2) For α and γ rationally independent the system (3.24) is well-known to be uniquely ergodic, see ref. 8. What we are going to demonstrate is that this happens for a positive measure set of parameters. To see this, first note that the velocity circle is determined by the two parameters N and E (see Eq. (3.9) and the discussion at the beginning of Section 3.2). Furthermore, it is possible to choose these two parameters from a positive measure set in $\mathbb{R}^2$ to ensure that $x(s) = v_t(s) > 0$ for all $s \in \mathbb{S}^1$. We keep the physical parameters (and thus γ) together with two integrals of motion, with N and E from the above set fixed and vary the third one, F from (3.10). What is shown below is that in this setting α depends on F in a continuous and strictly monotonic manner. This implies that α and γ can be rationally dependent only for countably many values of F, and thus by our choice of N and E the statement follows.

In the rest of the proof, all functions of s (e.g., $z(s), \beta(s), \alpha(s)$) refer to the original values of the integrals, while the primed ones (e.g., $z'(s), \beta'(s), \alpha'(s)$) indicate maps for F slightly increased. Since Lebesgue measure on the circle is invariant with respect to the transformation $s \to s_2$, we have

$$\alpha = \int_{\mathbb{S}^1} (\hat{\beta}(s) + \hat{\beta}(s_2)) ds = 2 \int_{\mathbb{S}^1} \hat{\beta}(s) ds.$$

Note that $x(s) = x'(s)$ and $y(s)$, $z(s)$ similarly, however as F is increased we get $w'(s) > w(s)$ for all $s \in \mathbb{S}^1$. This implies $\beta'(s) < \beta(s)$ (recall our convention on $x(s) > 0$) which yields, by a simple geometrical calculation, $\hat{\beta}'(s) < \hat{\beta}(s)$ for all $s \in \mathbb{S}^1$. Thus α depends on F strictly monotonically.

(3) A suspension flow above an ergodic map is ergodic iff the ceiling function is not cohomologous to a constant, see ref. 8. As $\tau(s)$ is C^∞ by Lemma 3.14 and γ is chosen to be Diophantine, we may apply Lemma 3.13 (note also Definition 3.10) to see that $\tau(s)$ is smoothly cohomologous to its average, a constant, thus the suspension flow is not ergodic. ▮

Remark 3.16. Note that an Anzai skew product (3.20) is never mixing as it has a factor which is a rotation by γ (with the semi-conjugacy being projection to the first $\mathbb{S}^1$-coordinate). On the other hand, in case the spectrum is not pure point only "horizontal" sets $B' = B \times \mathbb{S}^1$ do not mix weakly, see Lemma 3.12. To be more precise, if the spectrum is not pure point, restricting the unitary transformation of $\mathcal{L}^2(\mathbb{S}^1 \times \mathbb{S}^1)$ to the orthocomplement of the subspace generated by characteristic functions of horizontal sets, the spectrum is continuous (see ref. 1). Even though this phenomenon, if occurs at all in our system, can only occur for a zero measure set of parameters (in the nonDiophantine case), it seems to be an interesting point for further investigation.

4. TWO PARTICLES, ONE ROTATING SCATTERER

To close the paper we mention that our analysis also gives some insight to the system of *two* particles with only the inner scatterer rotating. The normal velocity components of the point particles are conserved, thus there are three velocities which evolve with the dynamics: the tangential velocity components of the point particles and the angular velocity of the inner scatterer. By conservation of full kinetic energy and angular momentum, as in the case of Section 3.2, the motion can be thought of as taking place on a velocity circle. Each of the two particles interacts with the inner scatterer: the interaction is realized as a reflection on the velocity circle.

At this point the exposition ceases to be parallel to that of Section 3.2. The two transformations may follow each other in a complicated manner and it is not enough to take two copies of the circle to describe the velocity evolution.

To describe the velocity dynamics, we consider a *two dimensional* dynamical system given by a natural Poincaré section, namely the first ball hitting the inner scatterer. The dynamics of the first coordinate, which is just the position on the velocity circle, is a piecewise isometry. It is either the reflection corresponding to the collision of the first ball at the inner scatterer, or the rotation corresponding to the composition of the collisions of both balls with the inner scatterer. Which one of these isometries takes place depends on the parity of the number of collisions the second particle makes between two consecutive collisions of the first particle. This information is coded by the second coordinate in a nonlinear way. The two dimensionality makes the analysis of this system complicated. None the less we are able to prove that, for a suitable choice of initial conditions (of positive measure), there is an arbitrary long time interval for which the second coordinate can be disregarded and for which the velocity motion follows a rotation orbit. The proof of this fact is not difficult, and could be checked by the interested reader. In particular, if the rotation angle is irrational, this rotation orbit becomes more and more dense on the velocity circle. Consequently the tangential velocity components of the individual point particles, which would be independent constants in the classical billiard setting, take values in sets that become more and more dense on the intervals allowed by the preserved quantities. Thus we can conclude that the rotor interaction introduced non-trivial coupling between the two point particles.

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